

A new proof of subcritical Trudinger-Moser inequalities on the whole Euclidean space

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Abstract

In this note, we give a new proof of subcritical Trudinger-Moser inequality on \mathbb{R}^n . All the existing proofs on this inequality are based on the rearrangement argument with respect to functions in the Sobolev space $W^{1,n}(\mathbb{R}^n)$. Our method avoids this technique and thus can be used in the Riemannian manifold case and in the entire Heisenberg group.

Key words: Trudinger-Moser inequality, Adams inequality

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1. Introduction

It was proved by Cao [4], Panda [9] and do Ó [5] that

Theorem A Let $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of the unit sphere in \mathbb{R}^n . Then for any $\alpha < \alpha_n$ there holds

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n} (|\nabla u|^n + |u|^n) dx \leq 1} \int_{\mathbb{R}^n} \left(e^{\alpha|u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx < \infty. \quad (1.1)$$

This result has various extensions, among which we mention Adachi and Tanaka [1], Ruf [11], Li-Ruf [7], Adimurthi-Yang [3]. To the authors' knowledge, all the existing proofs of such an inequality are based on rearrangement argument with respect to functions in the Sobolev space $W^{1,n}(\mathbb{R}^n)$. The purpose of this short note is to provide a new method to reprove Theorem A. Namely, we use a technique of the analogy of unity decomposition. More precisely, for any $u \in W^{1,n}(\mathbb{R}^n)$, we first take a cut-off function $\phi_i \in C_0^\infty(B_R(x_i))$ such that $0 \leq \phi_i \leq 1$ on $B_R(x_i)$, $\phi_i \equiv 1$ on $B_{R/2}(x_i)$. Then, using the usual Trudinger-Moser inequality [8, 10, 13] for bounded domain, we prove a key estimate

$$\int_{\mathbb{R}^n} \left(e^{\alpha|\phi_i u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |\phi_i u|^{\frac{nk}{n-1}}}{k!} \right) dx \leq \int_{\mathbb{R}^n} |\nabla(\phi_i u)|^n dx \quad (1.2)$$

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under the condition that

$$\int_{\mathbb{R}^n} |\nabla(\phi_i u)|^n dx \leq 1.$$

The power of (1.2) is evident. It permits us to approximate u by $\sum_i \phi_i u$, where every ϕ_i is supported in $B_R(x_i)$, $\mathbb{R}^n = \cup_{i=1}^{\infty} B_{R/2}(x_i)$, and any fixed $x \in \mathbb{R}^n$ belongs to at most $c(n)$ balls $B_R(x_i)$ for some universal constant $c(n)$. If we further take ϕ_i such that $|\nabla \phi_i| \leq 4/R$. Note that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$\int_{\mathbb{R}^n} |\nabla(\phi_i u)|^n dx \leq (1 + \epsilon) \int_{\mathbb{R}^n} |\nabla u|^n dx + \frac{C(\epsilon)}{R^n} \int_{\mathbb{R}^n} |u|^n dx.$$

Selecting $\epsilon > 0$ sufficiently small and $R > 0$ sufficiently large, we get the desired result.

Similar idea was used by the first named author to deal with similar problems on complete Riemannian manifolds [14] or the entire Heisenberg group [16]. Note that due to the complicated geometric structure, we have not obtained Theorem A on manifolds, but a weaker result. Namely

Theorem B *Let (M, g) be a complete noncompact Riemannian n -manifold. Suppose that its Ricci curvature has lower bound, namely $\text{Rc}_{(M,g)} \geq Kg$ for some constant $K \in \mathbb{R}$, and its injectivity radius is strictly positive, namely $\text{inj}_{(M,g)} \geq i_0$ for some constant $i_0 > 0$. Then we have*

(i) *for any $0 \leq \alpha < \alpha_n$ there exists positive constants τ and β depending only on n, α, K and i_0 such that*

$$\sup_{u \in W^{1,n}(M), \|u\|_{1,\tau} \leq 1} \int_M \left(e^{\alpha|u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!} \right) dv_g \leq \beta, \quad (1.3)$$

where

$$\|u\|_{1,\tau} = \left(\int_M |\nabla_g u|^n dv_g \right)^{1/n} + \tau \left(\int_M |u|^n dv_g \right)^{1/n}. \quad (1.4)$$

As a consequence, $W^{1,n}(M)$ is embedded in $L^q(M)$ continuously for all $q \geq n$;

(ii) for any $\alpha > \alpha_n$ and any $\tau > 0$, the supremum in (1.3) is infinite;

(iii) for any $u \in W^{1,n}(M)$ and any $\alpha > 0$, the integrals in (1.3) are still finite.

We say more words about this method. For Sobolev inequalities on complete noncompact Riemannian manifolds, unity decomposition was employed by Hebey et al. [6]. In the case of Trudinger-Moser inequality, it is not evidently applicable. We are lucky to find its analogy ([14], Lemma 4.1).

2. Preliminary lemmas

We first give a local estimate concerning the Trudinger-Moser functional. Precisely we have

Lemma 1 *For any $x_0 \in \mathbb{R}^n$ and any $u \in W_0^{1,n}(B_R(x_0))$, $\int_{B_R(x_0)} |\nabla u|^n dx \leq 1$, we have*

$$\int_{B_R(x_0)} \left(e^{\alpha_n |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx \leq C(n) R^n \int_{B_R(x_0)} |\nabla u|^n dx, \quad (2.1)$$

where $C(n)$ is a constant depending only on n .

Proof. Essentially this is the same as ([14], Lemma 4.1). For reader's convenience we give the details here. It is well known [8, 10, 13] that

$$\sup_{u \in W_0^{1,n}(B_R(x_0)), \int_{B_R(x_0)} |\nabla u|^n dx \leq 1} \int_{B_R(x_0)} e^{\alpha_n |u|^{\frac{n}{n-1}}} dx \leq C(n) R^n. \quad (2.2)$$

Letting $\tilde{u} = \frac{u}{\|\nabla u\|_{L^n(B_R(x_0))}}$ for any $u \in W_0^{1,n}(B_R(x_0)) \setminus \{0\}$, we have

$$\begin{aligned} \int_{B_R(x_0)} \left(e^{\alpha_n |\tilde{u}|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |\tilde{u}|^{\frac{nk}{n-1}}}{k!} \right) dx &\geq \frac{1}{\|\nabla u\|_{L^n(B_R(x_0))}} \int_{B_R(x_0)} \sum_{k=n-1}^{\infty} \frac{\alpha_n^k |u|^{\frac{nk}{n-1}}}{k!} dx \\ &= \frac{1}{\|\nabla u\|_{L^n(B_R(x_0))}} \int_{B_R(x_0)} \left(e^{\alpha_n |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx. \end{aligned} \quad (2.3)$$

Combining (2.2) and (2.3), we get the desired result. \square

Also we need a covering lemma of \mathbb{R}^n , see for example ([6], Lemma 1.6).

Lemma 2 For any $R > 0$, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ such that

- (i) $\cup_{i=1}^{\infty} B_{R/2}(x_i) = \mathbb{R}^n$;
- (ii) $\forall i \neq j, B_{R/4}(x_i) \cap B_{R/4}(x_j) = \emptyset$;
- (iii) $\forall x \in \mathbb{R}^n$, x belongs to at most N balls $B_R(x_i)$ for some integer N .

3. Proof of Theorem A

We shall obtain a global inequality (1.1) by gluing local estimates (2.1).

Proof of Theorem A. Let $R > 0$ to be determined later. Let ϕ_i be the cut-off function satisfies the following conditions: (i) $\phi_i \in C_0^{\infty}(B_R(x_i))$; (ii) $0 \leq \phi_i \leq 1$ on $B_R(x_i)$ and $\phi_i \equiv 1$ on $B_{R/2}(x_i)$; (iii) $|\nabla \phi_i(x)| \leq 4/R$. For $u \in W^{1,n}(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} (|\nabla u|^n + |u|^n) dx \leq 1, \quad (3.1)$$

we have $\phi_i u \in W_0^{1,n}(B_R(x_i))$, using Cauchy inequality with ϵ term we obtain

$$\begin{aligned} \int_{B_R(x_i)} |\nabla(\phi_i u)|^n dx &\leq (1 + \epsilon) \int_{B_R(x_i)} \phi_i^n |\nabla u|^n dx + C(\epsilon) \int_{B_R(x_i)} |\nabla \phi_i|^n |u|^n dx \\ &\leq (1 + \epsilon) \int_{B_R(x_i)} |\nabla u|^n dx + \frac{C(\epsilon)}{R^n} \int_{B_R(x_i)} |u|^n dx \\ &\leq (1 + \epsilon) \int_{B_R(x_i)} (|\nabla u|^n + |u|^n) dx, \end{aligned} \quad (3.2)$$

where in the last inequality we choose a sufficiently large R to make sure $\frac{C(\epsilon)}{R^n} \leq (1 + \epsilon)$. Let $\alpha_\epsilon = \frac{\alpha_n}{(1+\epsilon)^{1/(n-1)}}$ and $\widetilde{\phi_i u} = \frac{\phi_i u}{(1+\epsilon)^{1/n}}$. Noting that $\widetilde{\phi_i u} \in W_0^{1,n}(B_R(x_i))$, we have by (3.2) and Lemma 1

$$\begin{aligned}
\int_{B_{\frac{R}{2}}(x_i)} \left(e^{\alpha_\epsilon |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx &\leq \int_{B_R(x_i)} \left(e^{\alpha_\epsilon |\phi_i u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |\phi_i u|^{\frac{nk}{n-1}}}{k!} \right) dx \\
&= \int_{B_R(x_i)} \left(e^{\alpha_n |\widetilde{\phi_i u}|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |\widetilde{\phi_i u}|^{\frac{nk}{n-1}}}{k!} \right) dx \\
&\leq C(n) R^n \int_{B_R(x_i)} |\nabla(\widetilde{\phi_i u})|^n dx \\
&\leq C(n) R^n \int_{B_R(x_i)} (|\nabla u|^n + |u|^n) dx.
\end{aligned} \tag{3.3}$$

By Lemma 2 and (3.3), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(e^{\alpha_\epsilon |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx &\leq \int_{\bigcup_{i=1}^\infty B_{\frac{R}{2}}(x_i)} \left(e^{\alpha_\epsilon |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx \\
&\leq \sum_{i=1}^\infty \int_{B_{\frac{R}{2}}(x_i)} \left(e^{\alpha_\epsilon |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{\frac{nk}{n-1}}}{k!} \right) dx \\
&\leq \sum_{i=1}^\infty C(n) R^n \int_{B_R(x_i)} (|\nabla u|^n + |u|^n) dx \\
&\leq C(n) R^n N \int_{\mathbb{R}^n} (|\nabla u|^n + |u|^n) dx \\
&\leq C(n) R^n N.
\end{aligned} \tag{3.4}$$

For any $\alpha < \alpha_n$, we can choose $\epsilon > 0$ sufficiently small such that $\alpha < \alpha_\epsilon$. This ends the proof of Theorem A. \square

4. Concluding remarks

Using the same idea to prove Theorem A, we can also prove the subcritical Adams inequality in \mathbb{R}^n [2, 12, 15], which strengthen ([14], Theorem 2.6). Since the proof is completely analogous to our proof of Theorem A, we leave it to the reader.

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